Linear Reductive Groups Talk

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So the idea of the classification is that from a reductive group we will get a root datum, and these root datum uniquely determine the reductive group. This construction essentially tells us that a reductive group is determined only by its maximal torus the characters of that torus.

Theorem ([BC79] Springer 2.9(i))**.** *For any root datum Ψ with reduced root system there is a connected reductive group G and a maximal torus T such that $\Psi = \psi(G,T)$. The pair (G,T) are unique up to isomorphism.*

1 Defining A Group Scheme

Reference is the introduction to EGA 1971. We will only consider affine (group) schemes and varieties. I think functors from algebras to groups in the way of GL, Sp, Sl etc are a natural thing to consider, after all GL seems sort of intrinsic as a structure aside from whatever ring you want to plug in. What I want to motivate is why they are representable. Essentially they are the matrix groups defined by polynomials.

To see this we consider an abstract variety.

1.1 Varieties

The minimum algebraic structure to talk about polynomials is a ring, k. We are multiplying and adding the variables. Given such a polynomial the minimal algebraic structure for which it makes sense to put values into the polynomial is a k-algebra. We need to be able to multiply and add values in k.

Algebra is the study of rings and therefore the study of polynomimals. Geometry is the study of curves. Therefore algebaric geometry is the study of curves comming from polynomials. One way curves come from polynomials is by considering their zero sets.

This is the principle object then of algebraic geometry.

Mathemtaically this might be formulated as by letting k be a commutative ring with unit and A a k-algebra. Let $P_I = P \;\; := \;\; k[(T_i)_{i\in I}]$ be the polynomial ring in some number of variables, and for every $t = (t_i)_{i\in I} \in A^I$ denote the ring homomorphism $P \to A$

$$ev_t : T_i \mapsto t_i$$

simply by $F \mapsto F(t)$.

Now given a collection of polynomials $(F_j)_{j\in J} \in P^J$ we want to study all the $t \in A^I$ such that for every $j \in J$

$$F_j(t) = 0$$

So given the above definition it is clear that we are studying the assignment to each polynomial its set of zeroes. We claim only that this assignemnt is functorial.

Theorem. *The map*

$$V_{(F_j)_j} : k - Alg \to Set$$

sending objects $A \mapsto \{t \in A^I : \forall j \in J \;\; F_j(t) = 0\}$ *and morphisms to the restriction of their product* $\varphi \mapsto \varphi^I|_{V_{(F_j)}(A)}$ *defines a functor.*

Proof. This is immediate.□

If we want to consider the gemoetry of these varieties, not their specific values but how they "look" one might ask when are two of these functors isomorphic. Here we claim that up to isomorphism these functors are all representable.

First it is clear that if $\mathfrak{J} = ((F_j)_j)$ is the ideal generated by the polynomials then we actually have equality of functors $V_{\mathfrak{J}} = V_{(F_j)_j}$.

Theorem.

$$V_{\mathfrak{J}} \cong \mathrm{Hom}_k(P_I/\mathfrak{J}, -)$$

Proof. Recall that an isomorphism of functors is a natural transformation, α, such that each of the maps $\alpha(A) \in \mathrm{Hom}_{Set}(V_{\mathfrak{J}}(A), \mathrm{Hom}_k(P_I/\mathfrak{J}, A))$ are isomorphisms, i.e. bijections.

First we claim that the isomorphism is true for the zero polynomial. In this case we want to show that for every k-algebra A we have a natural bijection

$$ev(A) : A^I \cong \mathrm{Hom}_k(P_I, A)$$

The map $t \mapsto ev_t$ suffices.

sub-proof. k-algebra homomorphisms are determined by where the indeterminants of the polynomial ring are sent. Hence the map $\mathrm{Hom}_k(P_I, A) \to A^I$ given by $f \mapsto (f(T_i))_{i\in I} \in A^I$ is a bijection and its inverse is the evaluation map.

Naturality follows from the fact that evaluation is a homomorphism.

■

Now the general $V_{\mathfrak{J}}$ is a subfunctor of the above case. Applying the isomorphism in the zero case we get

$$ev(A)(V_{\mathfrak{J}}(A)) = \{f \in Hom_k(P_I, A) : f(\mathfrak{J}) = 0\} \cong \mathrm{Hom}_k(P_I/\mathfrak{J}, A)$$

Where the second equality is the universal property of the quotient. This shows the bijection. Naturality follows from the zero case as well.

□

1.2 Group Schemes

An (affine) group scheme is then simply a representable functor

$$\mathrm{Hom}_k(A, -) : k - Alg \to Groups$$

we can see how this is an abstract variety, and for this reason we narrow in on these functors between algebras and sets. We therefore have a category with these as objects and natural transformations as our morphisms. Following [Mak] we restrict to the case where our affine group scheme is of finite type and reduced, i.e. represented by a finitely generated k algebra with no nilpotent elements. This is what is called an algebraic group and we do this to simplify everything.

We also mention the important type of function between group schemes, that of isogeny, that is a surjective map with finite kernel. If there exists an isogeny $G \to H$ then we say G is isogenous to H, note however that this is not symetric.

1.3 Examples

There are many examples in [Spr98][Mil17][Milb][Mila][Mak].

Example 1 $\mathbb{G}_m$**:**

Consider the representable functor

$$\mathbb{G}_m \quad := \quad \mathrm{Hom}_K(K[x, y]/(xy - 1), -)$$

These are ring maps that are K linear. Because $y = x^{-1}$ we know that $f(y) = f(x^{-1}) = f(x)^{-1}$ for $f \in \mathbb{G}_m(R)$. Thus the maps are determined by where they send x, moreover they always send it to something invertible, i.e. $Imf \subseteq R^{\times}$. For each element $r \in R^{\times}$ we also have a map sending $x \to r$ hence there is an isomorphism (of sets) that allows us to unduce a group structure.

This proves two things: that this is a group scheme and it always gives the group of multiplicative units.

Note that morphisms are sent to morphisms because the are already K-algebra homs and Hom just pullsback/pushforward.

Example 2 Gl_n**:**

Consider the representable functor

$$GL_n \quad := \quad \mathrm{Hom}(K[x_{i,j} : 1 \leq i, j \leq n][y]/(y \det(x_{i,j}) - 1), -)$$

For the same reason as above a morphism here would be a choice of $x_{i,j}$ for each $1 \leq i, j \leq n$, i.e. a matrix of size $n \times n$, and a choice of y. The ideal we have modded out by tells us that the choice for y must multiply the determinant to 1, this means y is actually fixed by the determinant and that the determinant must be invertible. Hence we are restricted to the matricies whose determinants are units. Thus the image is isomorphic as sets to $GL_n(R)$ and hence we can induce a group structure.

Example 3 Sp_4**:**

In classical mathematics, we would say that $\mathrm{Sp}_4(K)$ is the group of matricies such that $A \in \mathrm{Sp}_4(K)$ implies that $A^T M A = M$ where

$$M = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

(this matrix is chosen for convenience). Lets go the other direction and try to make a functor that has R points the matricies that have this property.

First if $A = (a_{i,j})_{0 \le i,j \le 3}$

$$\begin{aligned} A^T M A = M &\iff \sum_{0 \le j,k \le 3} a_{j,i} m_{j,k} a_{k,l} = a_{i,l} \quad \forall 0 \le i,l \le 3 \\ &\iff \sum_{0 \le j \le 3} a_{j,i} m_{j,4-j-1} a_{4-j-1,l} = a_{i,l} \quad \forall 0 \le i,l \le 3 \\ &\iff a_{0,i} a_{3,l} - a_{1,i} a_{2,l} + a_{2,i} a_{1,l} - a_{3,i} a_{0,l} = a_{i,l} \quad \forall 0 \le i,l \le 3 \end{aligned}$$

So these are 4×4 matricies satisfying the relation above. i.e. choices of 16 elements of R such that $a_{0,i} a_{3,l} - a_{1,i} a_{2,l} + a_{2,i} a_{1,l} - a_{3,i} a_{0,l} - a_{i,l} = 0$. Hence this is in bijection with

$$\mathrm{Hom}_K(K[x_{ij} : 0 \le i,j \le 3]/(a_{0,i} a_{3,l} - a_{1,i} a_{2,l} + a_{2,i} a_{1,l} - a_{3,i} a_{0,l} - a_{i,l}), -)$$

which we "pushforward" the group structure to.

2 The Idea of Reductive

We combine both [GH24] and [Mak].

First of all we consider K an algebraically closed field. We let G be an algebraic group defined over K. Not being algebraically closed introduces unelightening technicalities, moreover most of the properties are checked by base changing to the algebraic closure which will agree on points, thus we lose little restricting to this case. The main difference is that the split tori will change.

2.1 Linear Groups

A representations of a group is then a morphism

$$G \to GL_V$$

we moreover say that this is a faithful representation if it is a closed immersion i.e. the map of algebras is surjective (inverse of Yonedas lemma). If $X \to Y$ is a closed immersion then for every R the map $X(R) \to Y(R)$ is injective.

A linear group is then one such that there exists a faithful representation.

2.2 Connected

We say G is connected if in the algebra representing it the only idempotents (square to themselves) are 0 and 1.

We remark that this agrees with the $\mathbb{C}$ points of G (when they are defined) being connected in the induced "analytic" topology.

2.3 Normal

A subgroup is normal inside G if the K points form a normal subgroup of $G(K)$.

2.4 Unipotent

Let $x \in GL(K)$, then we call x semi-simple if it is diagonalizable, nilpotent if there is some m for which $x^m = 0$ and unipotent if $x - I$ is nilpotent.

From the Jordan decomposition over an algebraically closed field we know that every x can be uniquely written as the product of a semisimple and unipotent part, denoted x_s, x_u respectively. For instance if we have a Jordan block

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix}$$

where clearly

$$\begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix} - I = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix}$$

is nilpotent.

In a linear group we have the same decomposition of its K points, such that it agrees with the decomposition of its image under any embedding. i.e. if $\varphi : G(K) \to GL_V(K)$ is an embedding

$$\varphi(x_s) = \varphi(x)_s \text{ and } \varphi(x_u) = \varphi(x)_u$$

Moreover this decomposition is preserved by any morphism of groups. We then say that an element of the K points, $x \in G(K)$, isunipotent if $x = x_u$. This defines a subgroup G_u of unipotent elements of the K points. We say that G is unipotent when $G(K) = G_u$ (note that we can do this only because we restricted to reduced and finite type implying that G is smooth).

2.5 Reductive Group

The unipotent radical of a group is then the largest connected normal unipotent subgroup of G. G is reductive if the unipotent radical is trivial.

So reductive suggests that we are kind of removing some of the unipotent parts and hence more of the matricies should be diagonalisable or indeed that the representation theory should be totally reducible.

3 Talk 2

3.1 Recall

Last time I defined what a connected reductive linear algebraic group over an algebraically closed field is. From now on I refer to these as GROUPS over k.

- Linear algebraic: An affine variety. i.e. representable by an algebra with no nilpotents and finitely generated. (which implies linear)
- Connected: Algebra has no non-trivial idempotents.
- Reductive: Trivial nilradical.

3.2 Examples

The proofs for GL_n and Sp are non-trivial and would require us to develop a small amount of theory before they became tractable in the time frame of a lecture. The idea is that the unipotent radical is the unipotent part of the radical, the radical is the identity component of the intersection of all the borels. Gl is connected and the borels are upper and lower triangular matricies. Thus the unipotent radical is unipotent diagonal matricies which is just the identity. Other matrix groups are similar (at least in Makisumi).

Example 1 $\mathbb{G}_a$**:**

This is an example of a group which is not reductive. We define it as $\mathbb{G}_a(R) \ := \ \mathrm{Hom}(K[x], R)$, which is just a choice of element in R, which is an additive group.

We need to show that it is linear. A representation is a morphism $\mathbb{G}_a \to GL_V$, or more concretely a morphism of groups for every $R \in k-alg$ $\mathbb{G}_a(R) \to GL_n(R)$ such that some diagrams commute. We require them to be injective. Notice that

$$\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ & 1 \end{pmatrix}$$

Hence inclusion into the top right corner is a clearly injective group homomorphism. Concretely then we define

$$\sigma(R) : \mathrm{Hom}(K[x], R) \to \mathrm{Hom}(K[x_{i,j} : 1 \leq i, j \leq n][y]/(y \det(x_{i,j}) - 1), R)$$

$$ev_r \mapsto ev_{(1,r,0,1)}$$

We need to show that the morphism

$$f : \mathcal{O}(GL_n) = K[x_{i,j} : 1 \leq i, j \leq n][y]/(y \det(x_{i,j}) - 1) \to \mathcal{O}(\mathbb{G}_a) = K[x]$$

is surjective. We know that $\mathrm{Hom}(f, -) = \sigma$ and so if $g \in \mathrm{Hom}(K[x], R)$ then $\mathrm{Hom}(f, -)(g) = \sigma(g) = g \circ f$, hence we have

$$ev_{1,r,0,1} = ev_r \circ f$$

so $f = ev_{1,x,0,1}$ which is a clear surjection.

It may be the case (by Jonah) that injectivity on points is sufficient in the nice case of algebraic groups over algebraically closed field.

We remark that any algebraic group in our sense is linear.

Now the K points are then K^+ or equivilently the subgroup of GL_2 given by $\left\{ \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} : a \in K \right\}$.

Then because the unipotent parts will agree on any embedding we can see that this whole group is unipotent. It is trivially normal inside itself so if it is connected then it is its own unipotent radical. Indeed the idempotents of $K[x]$ are those of K i.e. only 1 and 0, so it is connected.

Thus this group is its own unipotent radical, moreover it is far from trivial and hence this is not a reductive group.

4 The Category of Root Datum

Following [BC79][Springer]. An abstract root datum is a quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ such that

- X and $X^\vee$ are free abelian groups of finite type.
- There exists a function $\langle , \rangle : X \times X^\vee \to \mathbb{Z}$
- Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$ respectively
- There is a bijection between Φ and $\Phi^\vee$, which we denote $(-)^\vee$
- For every $\alpha \in \Phi$ we have $\langle \alpha, \alpha^\vee \rangle = 2$
- For every $\alpha \in \Phi$ the function $s_\alpha : X \to X$

  $$x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$$

  fixes Φ i.e. $s_\alpha(\Phi) \subseteq \Phi$
- For every $\alpha^\vee \in \Phi^\vee$ the function $s_{\alpha^\vee} : X^\vee \to X^\vee$

  $$x \mapsto x - \langle \alpha, x \rangle \alpha^\vee$$

  fixes $\Phi^\vee$

For those who understand if you tensor this over $\mathbb{Z}$ with $\mathbb{R}$ then you get a root system from the classification of Lie algebras. I assume that the motivation for these aximos comes from this as well and so I neglect it because I dont know it. At the very least the data seems more combinatorial and so we can imagine they might be easier to work with than our unwhiley reductive groups.

We make the collection of such root datum into a category by specifying morphisms as isogenies. In particular if $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ and $\Psi' = (Y, \Delta, Y^\vee, \Delta^\vee)$ are two root datum then a map $f : X \to Y$ is an isogeny if

- f is a homomorphism of groups
- f is injective
- The image of f has finite index in y
- f restricts to a bijection $\Phi \to \Delta$
- The transpose of f, f^t, restricts to a bijection $\Delta^\vee \to \Phi^\vee$

The idea is now that there is an equivilence of categories between this and the category of GROUPS over k.

5 Constructing a Root Datum out of a Group

We now explain how to get a root datum from an (affine algebraic) reductive group over an algebraically closed field. The references are [Spr98], [GH24] and [Mak]. We will routinely abuse Yonedas lemma and make use of the equivilence of categories that it provides.

5.1 Tori

Lemma.

$$\mathrm{Hom}_{Group^{k-alg}}(\mathbb{G}_m, \mathbb{G}_m) \overset{\varphi}{\cong} \mathbb{Z}$$

as groups (the group structure on the Hom is pointwise).

> ***Proof.*** Requires the theory of Hopf algebras... or general theory from Milne to prove. Basically $\varphi(x) = x^n$ for some $n \in \mathbb{Z}$ always. Some pretty explicit stuff in [Not]

$$\begin{array}{ccc}
\mathrm{Hom}_{Group^{k-alg}}(G,H) & \xrightarrow{\quad\cong\quad} & \mathrm{Hom}_{Hopf}(\mathcal{O}(H),\mathcal{O}(G)) \\
\big\downarrow & \searrow & \big\downarrow \\
\mathrm{Hom}_{Set^{k-alg}}(G,H) & \xrightarrow{\quad\cong\quad} & \mathrm{Hom}_k(\mathcal{O}(H),\mathcal{O}(G))
\end{array}$$

A torus of a group is a subgroup that is isomorphic to $\mathbb{G}_m^n$ for some $n \in \mathbb{N}$. This nomenclature might make sense becuase $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ which is a punctured disc, homotopic to a circle and tori and just circles bro. Moreover there is a similarity with Tori in Lie groups that are actually tori, but have algebraic characterisations.

A maximal torus is a torus contained in no other tori.

Lemma (Grothendieck, Borel, Tits)**.** *Every reductive group contains a maximal torus and any two maximal tori are conjugate.*

From now on we fix a maximal torus $T \leq G$. Chenyan claims that the root datum are isomorphic.

source

Sp:

Recall the definition we gave $\mathrm{Sp}_4(K)$ is the group of matricies such that $A \in \mathrm{Sp}_4(K)$ implies that $A^T M A = M$ where

$$M = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

or given as a functor

$$\mathrm{Hom}_K(K[x_{ij} : 0 \leq i,j \leq 3]/(a_{0,i}a_{3,l} - a_{1,i}a_{2,l} + a_{2,i}a_{1,l} - a_{3,i}a_{0,l} - a_{i,l}), -)$$

That this is linear is clear by construction, connected is beleivable from the relation that we mod out, reductive is not obvious again without developing a bit more theory.

From our knowledge of linear algebra we get

$$\mathrm{Sp}_4 \subseteq \mathrm{Sl}_4 \subseteq \mathrm{Gl}_4$$

and so we can compute a torus of Sp_4 from the standard one of Sl, namely $D =$ diagonal matricies in Sl_4 [MT][Mak] say that this is fine.:

$$T = D \cap \mathrm{Sp}_4$$

sufficient condition and proof for this to be the case

A diagonal matrix commutes with M iff

$$diag(t_1,t_2,t_3,t_3)Mdiag(t_1,t_2,t_3,t_3) = \begin{pmatrix} & & & t_1 \\ & & -t_2 & \\ & t_3 & & \\ t_4 & & & \end{pmatrix} diag(t_1,t_2,t_3,t_4) = \begin{pmatrix} & & & t_1t_4 \\ & & -t_2t_3 & \\ & t_3t_2 & & \\ -t_4t_1 & & & \end{pmatrix} = M$$

hence $t_1t_4 = t_3t_2 = 1$ i.e. $t_1 = t_4^{-1}$ and $t_3 = t_2^{-1}$ thus

$$T = \left\{ diag(t_1, t_2, t_2^{-1}, t_1^{-1}) \right\}$$

This torus is maximal.

5.2 X and $X^\vee$

We now specify the first two peices of the root datum.

$$X^* \quad := \quad \mathrm{Hom}(T, \mathbb{G}_m), \quad X_* \quad := \quad \mathrm{Hom}(\mathbb{G}_m, T)$$

Lemma. *X^*, X_* are free abelian groups. In fact if $T \cong \mathbb{G}_m^n$ then they are both isomorphic as groups to $\mathbb{Z}^n$.*

Proof. X_* follows from the universal property of the product. The dual should be isomorphic (not rigerous)

When we compose $\alpha \in X^*, \beta \in X_*$ we get a function $\beta \circ \alpha : \mathbb{G}_m \to \mathbb{G}_m$ and so applying our isomorphism above we get a pairing

$$\langle , \rangle : X^* \times X_* \to \mathbb{Z}$$
$$\langle \alpha, \beta \rangle = \varphi(\beta \circ \alpha)$$

So this shows that these two satisfy all the properties of the $X, X^\vee$ in the root datum.

Sp:

This torus is clearly $T \cong \mathbb{G}_m^2$ and so the character groups are free abelian rank 2. Explicitly we have the bases:

$$\alpha_1 : diag(t_1, t_2, t_2^{-1}, t_1^{-1}) \mapsto t_1$$
$$\alpha_2 : diag(t_1, t_2, t_2^{-1}, t_1^{-1}) \mapsto t_2$$

for $\mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ and

$$\lambda_1 : x \mapsto \begin{pmatrix} x & & & \\ & 1 & & \\ & & 1 & \\ & & & x^{-1} \end{pmatrix}$$

$$\lambda_2 : x \mapsto \begin{pmatrix} 1 & & & \\ & x & & \\ & & x^{-1} & \\ & & & 1 \end{pmatrix}$$

for $\mathrm{Hom}(\mathbb{G}_m, T) \cong \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$. By the way these follow from the general results above that I didnt prove, but they also are somewhat intuitive (it is clear that these are maps, its not clear that they are the only possible maps).

5.3 Φ

The hard part is now to find subgroups of these that will satisfy the many more properties imposed by the root datum. To get a subgroup of X^* we will go via the Lie algebra of G [Mila][XI. 7] has the most detail.

Lie Algebra

Consider the map of K-modules

$$\pi_R : R[t]/(t^2) \cong R \oplus R\epsilon \to R$$

$$a + b\epsilon \to a$$

where ϵ is some symbol such that $\epsilon^2 = 0$, i.e. an infinitesimal. This smells of looking at the tangent vectors at the identity, i.e. the Lie algebra of a Lie group.

We define $\mathfrak{g}(R) := kerG(\pi_R)$. Note that the bracket is not really important for us, however its can be defined either cannonically or explicitly. Moreover $K[t]/(t^2)$ is natrually a K vector space, so we have an arrow $\rho_r : K[t]/(t^2) \to K[t]/(t^2)$ for every $r \in K$ which induces a K vector space structure on $G(K[t]/(t^2))$ by $\bar{\rho}_k : G(K[t]/(t^2)) \to G(K[t]/(t^2)), \bar{\rho}_r = G(\rho_r)$.

Sp:

Someone computed the Lie algebra once and it was matricies of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & -a_{13} \\ a_{31} & a_{32} & -a_{22} & a_{12} \\ a_{41} & -a_{31} & a_{21} & -a_{11} \end{pmatrix}$$

Adjoint Representation

A representation of G on the Lie algebra we will define to be a morphism of groups

$$G \to GL_{\mathfrak{g}} := Aut(\mathfrak{g}(R))$$

It is clear that

$$R \hookrightarrow R[t]/(t^2)$$

which tells us that

$$T(R) \hookrightarrow G(R) \hookrightarrow G(R[t]/(t^2))$$

it is also clear that $G(R[t]/(t^2))$ acts on itself by conjugation and hence we can restrict this to an action of $T(R)$ on $G(R[t]/(t^2))$ by conjugation.

Lemma. *This action preserves kernels.*

Hence this action becomes an action on $\mathfrak{g}(R)$ so we have specified the adjoint action of T on the Lie algebra of G, i.e. a morphism of algberaic groups $T \to Gl_{\mathfrak{g}}$.

check what this involves. Where did I get this from ?

By [Spr98][Theorem 3.2.3] or [Mila][Theorem XIV.4] we have the decomposition

$$\mathfrak{g} := \mathfrak{g}(K) = \bigoplus_{\alpha \in X^*} \mathfrak{g}_\alpha$$

where

$$g_\alpha = \{X \in \mathfrak{g} : \forall t \in T(K) \alpha(t) = t.X\}$$

where all but finitely many are non-zero.

Proof. Merely a sketch. We would need Hopf algebras again. First an affine group G is diagonalizable if and only if it is isomorphic to $\mathrm{Hom}_k(k[M], -)$ for some commutative group M (definition)(the group algebra) (it can be shown that for a finitely generated group this is a finite product of $\mathbb{G}_m$ and groups of roots of unity, hence the name).

Then $\mathbb{G}_m^n \cong \mathrm{Hom}_k(k[\mathbb{Z}^n], -)$. So its diagonalizable. In a diagonal group group the group like elements span its coordinate ring (as a k-vector space). But the group like elements of a an affine group represented by a group algebra are the group elements. Then the representation of the groups gives a co-module for the coordinare ring of G. We then show that our group elements span one dimensional subrepresentations and becuase they are a basis we have reduced the representation into one dimensional pieces.

The final piece is to see that $X^*(\mathrm{Hom}_k(k[\mathbb{Z}^n], -)) \cong X^*(\mathbb{G}_m^n) \cong \mathbb{Z}^n$ and so the peices are indeed indexed by the group corresponding to the coordinate ring, which is the same as the character group.

Finally we define

$$\Phi \quad := \quad \{\alpha \in X^* : \mathfrak{g}_\alpha \neq 0\}$$

Sp:

To find the roots we need to examine the adjoint action so we have

$$diag(x,y,y^{-1},x^{-1})\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & -a_{13} \\ a_{31} & a_{32} & -a_{22} & a_{12} \\ a_{41} & -a_{31} & a_{21} & -a_{11} \end{pmatrix}(diag(x,y,y^{-1},x^{-1}))^{-1} = \begin{pmatrix} a_{11} & a_{12}xy^{-1} & a_{13}xy & a_{14}x^2 \\ a_{21}x^{-1}y & a_{22} & a_{23}y^2 & -a_{13}xy \\ a_{31}(xy)^{-1} & a_{32}y^{-2} & -a_{22} & a_{12}xy^{-1} \\ a_{41}x^{-2} & -a_{31}(xy)^{-1} & a_{21}x^{-1}y & -a_{11} \end{pmatrix}$$

From general theory [GH24] we know that the weight spaces are of the form

$$\mathfrak{g}_\alpha = \{X \in G(K) : \forall T(K) \;\; Ad_t(X) = \alpha(t)X\}$$

So we want to solve

$$\begin{pmatrix} a_{11} & a_{12}xy^{-1} & a_{13}xy & a_{14}x^2 \\ a_{21}x^{-1}y & a_{22} & a_{23}y^2 & -a_{13}xy \\ a_{31}(xy)^{-1} & a_{32}y^{-2} & -a_{22} & a_{12}xy^{-1} \\ a_{41}x^{-2} & -a_{31}(xy)^{-1} & a_{21}x^{-1}y & -a_{11} \end{pmatrix} = x^n y^m \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & -a_{13} \\ a_{31} & a_{32} & -a_{22} & a_{12} \\ a_{41} & -a_{31} & a_{21} & -a_{11} \end{pmatrix}$$

Which by matching the powers of x, y tells us that $\Phi = \{\pm\alpha_i \pm \alpha_j : 1 \leq i, j \leq 2\}$ have eigenmatricies that are non-zero.

5.4 $\Phi^\vee$

We will define $\Phi^\vee$ via Φ and the Weil group. The details of this are in [Spr98] in particular chapter 3 and 7.

Now consider an element $\alpha \in \Phi$ we associate a torus T_α, the maximal torus of the kernel of α. This defines another split reductive group over K

$$G_\alpha \quad := \quad C_G(T_\alpha)$$

which contains T. This is all a black box, the quotient stuff is general theory but that the torus is still in there etc etc, I would like to know a source for this.

Sp:

There are eight roots. For each root we theoretically have to go through a whole process to compute the coroots. We will do this for one $\alpha_1\alpha_2$:

First

$$ker(\alpha_1\alpha_2) = \{diag(t_1, t_2, t_2^{-1}, t_1^{-1}) : (\alpha_1\alpha_2)(diag(t_1, t_2, t_2^{-1}, t_1^{-1})) = t_1t_2 = 1\} = \{diag(x, x^{-1}, x, x^{-1})\}$$

This is conveniently already a torus so we dont have to work so hard. Next we want its centralizer in G, that is we want to solve for what $g \in G(K)$ such that for all $x \in K^\times$ we have

$$diag(x, x^{-1}, x, x^{-1})^{-1} g diag(x, x^{-1}, x, x^{-1}) = g$$

i.e.

$$diag(x,x^{-1},x,x^{-1})^{-1}\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} diag(x,x^{-1},x,x^{-1}) = \begin{pmatrix} g_{11} & x^{-2}g_{12} & g_{13} & x^{-2}g_{14} \\ x^2g_{21} & g_{22} & x^2g_{23} & g_{24} \\ g_{31} & x^{-2}g_{32} & g_{33} & x^{-2}g_{34} \\ x^2g_{41} & g_{42} & x^2g_{43} & g_{44} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$$

but then for all x we must have $x^{\pm 2}b_{ij} = b_{ij}$ and therefore those entries must be zero. Hence the centraliser is matricies of the form

$$\begin{pmatrix} * & & * & \\ & * & & * \\ * & & * & \\ & * & & * \end{pmatrix}$$

Now indeed this looks like a reductive group (why not) and moreover we can see that yes in fact $T \leq C_G(T_\alpha)$ in our case.

Weil Group

The Weil group of G and T is

$$W(G,T)(K) \ := \ N_G(T)(K)/C_G(T)(K)$$

where N is the normalizer ($\{g \in G(K) : gT(K) = T(K)g\}$) and C is the centralizer ($\{\{g \in G(K) : \forall t \in T(K) \ \ gt = tg\}\}$). This also makes sense for any k/K field extension. We remark that this can be made into a scheme through the theory of GIT quotients, which I know nothing about. Then

$$W(G_\alpha, T)(K) = \langle s_\alpha \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

> ***Proof.*** [Spr98][7.1.4] If S is in the center of G then $W(R,T) \cong W(G/S, T/S)$. Our T_α is in the center of G_α and so
>
> $$W(G_\alpha, T) \cong W(G_\alpha/T_\alpha, T/T_\alpha)$$
>
> But $T/T_\alpha \cong \mathbb{G}_m$. This simplifies things, and we argue that the group has size 2.

This has a faithful representation as automorphisms of X^*. Let $\sigma = [n] \in W(G,T), \alpha \in X^*$ then $(\sigma.\alpha)(t) \ := \ \alpha(n^{-1}tn)$. We now conflate s_α with its image under this representation. Now we claim that there exists a unique $\alpha^\vee \in X_*$ such that

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$$

This simultaneously definies the set $\Phi^\vee$ and the bijection $\Phi \to \Phi^\vee$. The finiteness of Φ gaurentees the finiteness of $\Phi^\vee$.

What remains to check is the final three bullet points of the root datum axioms. We refer you to Springer, linear algebraic groups, for these proofs. Upon me skimming them once they appear to be somewhat elementary although tedious linear algebra and geometry type arguments, similar to those used in classifying root systems that you may have seen. Indeed several of them take place at the level of root systems.

Sp:

We failed to compute the Weil group for this example, so we will use an ansatze given by Makisumi. from which the root datum relations are obvious by direct computation.

Φ	$\Phi^\vee$
$2\alpha_1$	λ_1
$-2\alpha_1$	$-\lambda_1$
$2\alpha_2$	λ_2
$-2\alpha_2$	$-\lambda_2$
$\alpha_1 + \alpha_2$	$\lambda_1 + \lambda_2$
$-\alpha_1 + \alpha_2$	$-\lambda_1 + \lambda_2$
$\alpha_1 - \alpha_2$	$\lambda_1 - \lambda_2$
$-\alpha_1 - \alpha_2$	$-\lambda_1 - \lambda_2$

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